Much of microeconomic theory involves optimizing behavior. Households wish to maximize their total satisfaction from limited income and time. Producers wish to find the efficient combination of input services to produce each rate of output. Entrepreneurs wish to maximize profit. By specifying objectives as mathematical functions, we use **calculus** to find the values of independent or instrumental variables that generate the optimal value of that function. This appendix will present the calculus concepts that will allow me to use calculus terminology in presenting microeconomic topics, and will allow you to use calculus to follow numerical examples and solve the end-of-chapter problems. This appendix is not an adequate substitute for a formal calculus course. However, if you understand calculus, this chapter will help you to relate calculus to economics; if you have never taken a calculus course, this appendix my help convince you of the benefits of doing so. Most chapters have an appendix using calculus to develop the concepts in that chapter. While you will not need calculus to understand the concepts, if you understand calculus it will add depth and precision to your understanding.

Basic Concepts

Differential calculus, the topic of this appendix, is the algebra of change. In your principles of microeconomics course, you encountered many terms prefaced with the adjective *marginal*. Marginal cost is the cost of the last unit produced. Marginal revenue is the revenue of the last unit sold. Marginal utility is the satisfaction from the last unit consumed or purchased. Each to these concepts is *derived* from a mathematical relationship: cost depends on quantity produced, revenue depends on quantity sold, utility depends on the quantity purchased.

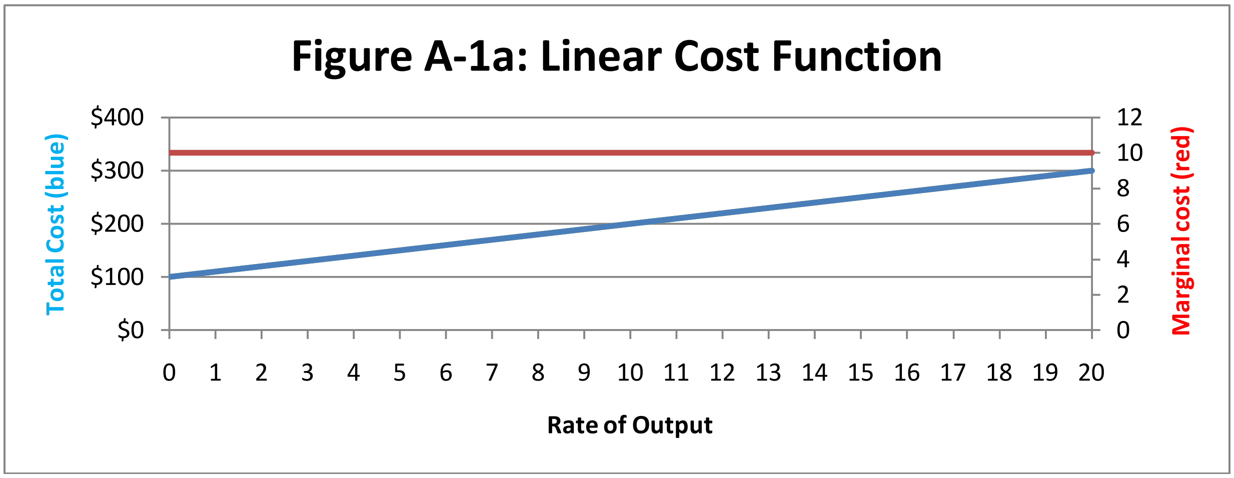
A **set** is simply a collection of items that follow some rule that distinguishes membership in the set from exclusion from the set. Typically, a variable of interest is drawn from a set. The firm's rate of output is drawn from the set of rates of production that fall within the firm's production capacity:

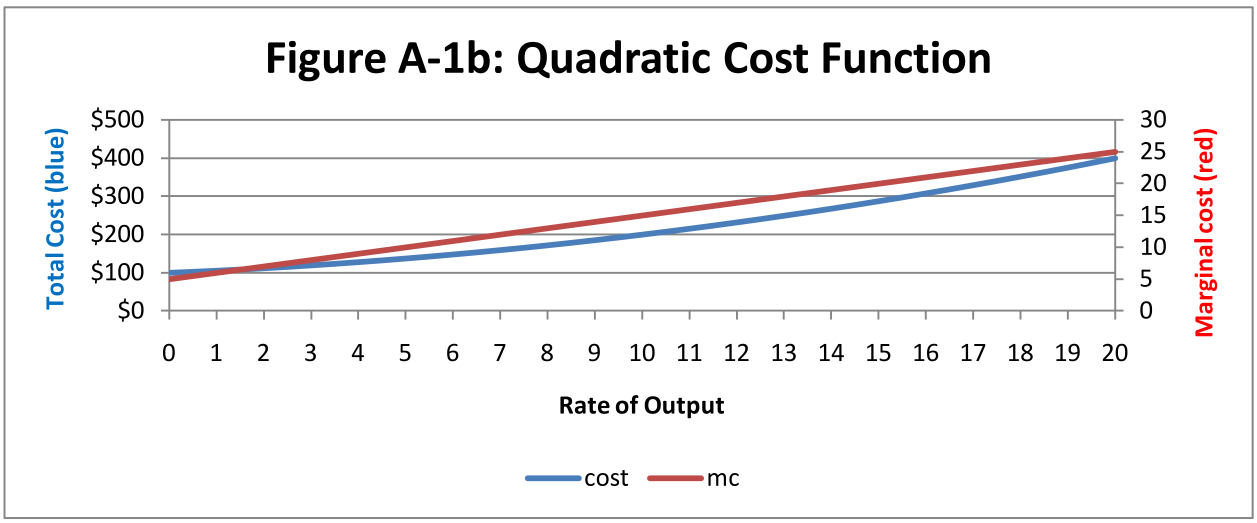
The household's consumption falls within a set of expenditures that satisfy the household's budget. An important set is the **set of ordered pairs**, each **element** or member of the set has two variables, the independent variable, or cause, is listed first, followed by the dependent variable or effect.[1] For instance, the price consumers are willing to pay depends on the quantity that a firm wishes to sell: $p = 100 - .5q$. A set of ordered pairs would be expressed as $p = \{(q,p) | p = \$100 - .5q\}$. What could change the role of the variables and make *quantity demanded* dependent on price: $q_d = \{(p,q_d) |\ q_d = 200 - 2p\}$. Which variable we treat as the cause and which we treat as the effect is often a matter of perspective. From the firm's point of view, the price the firm can charge depends on the amount offered for sale. From the household's viewpoint, the amount its members are willing to buy depends on the price charged. The set from which the first element of an ordered pair is drawn is called the **domain**. The set from which the second element of an ordered pair is drawn is called the **range.** Whether we describe this **relationship** as a willingness to pay (price as a function of output) or willingness to buy (quantity demanded as a function of price), each value of the independent variable implies only one value of the dependent variable; this type of relationship is called a **function**.

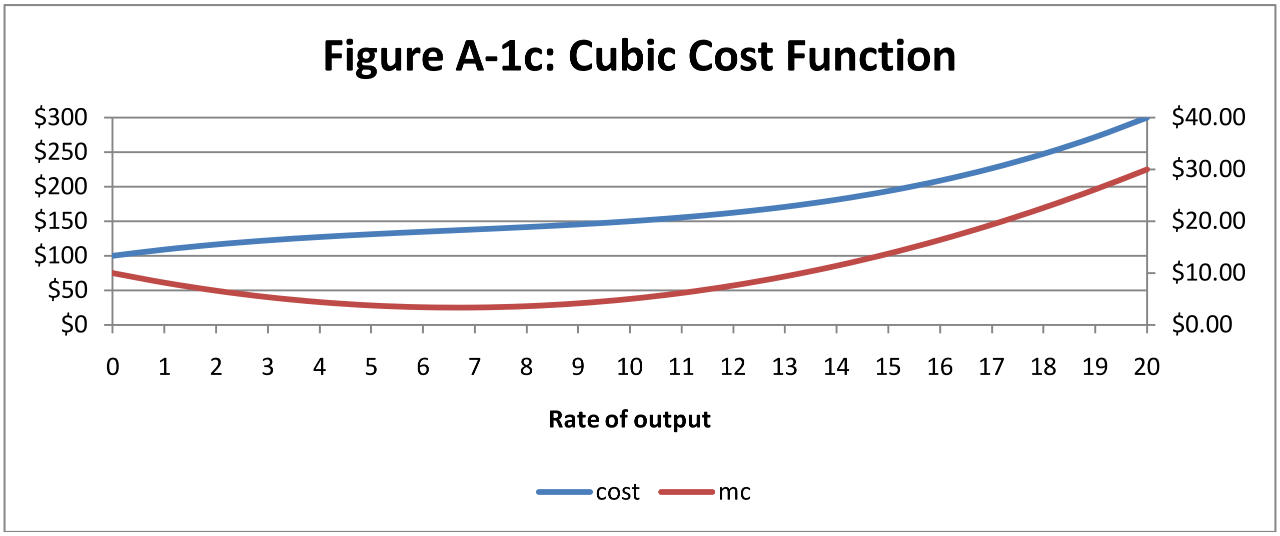
[1] Many economic functions have interchangeable independent and dependent variables. For instance, a demand typically treats quantity demanded as the dependent variable and price (along with other **exogenous variables**) as the independent variable. However, from the monopoly seller's perspective, quantity determines price. The consensus in economics is to plot quantity on the horizontal axis and price on the vertical axis, regardless of whether price is the dependent variable (in the price of the monopoly's average revenue curve – the inverse demand curve) or if price is the causal variable (as in the case of a individual buyers demand curve).

Marginal Analysis and Derivatives

Much of microeconomic theory is concerned with marginal analysis – determining how a change in the dependent variable is related to the value of the independent variable. *Marginal cost* is a function that expresses the cost of producing the last unit, which typically changes with the amount produced. Figure A-1a shows a linear total cost function, $C = a + bq$; marginal cost is equal to the constant slope of the function, b. In figure A-1b shows a quadratic cost function, $C = a + bq + cq^2$, so that marginal cost changes (systematically) with output: $MC = b + 2cq$. Figure A-1c shows a cubic total cost function of the form $C = a + bq + cq^2 + dq^3$; its marginal cost function is described by the *derivative* equation $MC = \frac{dC}{dq} = b + 2cq$. For each type of function, there is a rule for computing the slope of that function. We now turn to deriving the rules of derivative functions.







Deriving Derivatives

To find the slope of a linear function, we simply identify the common change in the dependent variable for each change of the independent variable. In Figure A-1a the total cost function is described by the equation $TC = 100 + 10q$. In Table A-1a we see that at each value of q, the slope of the equation is 10, whether the change in q is 1 unit, 2 units, or 20 units.

Table A-1a: Marginal Cost of Linear Cost Function

output	cost	C(q-1)	C(q) - C(q-1)	dC
0	100			10
1	110	100	10	10
2	120	110	10	10
3	130	120	10	10
4	140	130	10	10
5	150	140	10	10
6	160	150	10	10
7	170	160	10	10
8	180	170	10	10
9	190	180	10	10
10	200	190	10	10
11	210	200	10	10
12	220	210	10	10
13	230	220	10	10
14	240	230	10	10
15	250	240	10	10
16	260	250	10	10
17	270	260	10	10
18	280	270	10	10
19	290	280	10	10
20	300	290	10	10

In Table A-1b the total cost function is the quadratic function $C = 100 + 5q + ½\, q^2$. In Table A-1b we find that the change in cost is different for different values of output. To compute marginal cost, we allow the change in output to get progressively smaller; as the limit of the change in cost approaches 0, the estimated value of marginal cost gets closer to the formula, MC = 5 + q. The idea behind differential calculus is that as we take smaller changes in the independent variable, the slope of the line connecting those points on the graph of the function become progressively closer to the slope of the **tangent line** at the point in question.

Table A-1c shows the same calculations applied to the cost function depicted in Figure A-1c:

$$C = 100 + 10q - q^2 + 0.05q^3$$

In Table A-1c the formula for marginal cost is

— ______________________________ ,

which is an application of the polynomial rule of differentiation we develop in the next section.

Table A-1b: Quadratic Cost Function

output	cost	Δq	ΔC	ΔC/Δq	Δq	ΔC	ΔC/Δq	Δq	ΔC	ΔC/Δq	Δq	ΔC	ΔC/Δq	dC/dq
0	100.00	1	5.5	5.5	0.5	2.625	5.25	0.25	1.28125	5.125	0.1	0.505	5.05	5
1	105.50	1	6.5	6.5	0.5	3.125	6.25	0.25	1.53125	6.125	0.1	0.605	6.05	6
2	112.00	1	7.5	7.5	0.5	3.625	7.25	0.25	1.78125	7.125	0.1	0.705	7.05	7
3	119.50	1	8.5	8.5	0.5	4.125	8.25	0.25	2.03125	8.125	0.1	0.805	8.05	8
4	128.00	1	9.5	9.5	0.5	4.625	9.25	0.25	2.28125	9.125	0.1	0.905	9.05	9
5	137.50	1	10.5	10.5	0.5	5.125	10.25	0.25	2.53125	10.125	0.1	1.005	10.05	10
6	148.00	1	11.5	11.5	0.5	5.625	11.25	0.25	2.78125	11.125	0.1	1.105	11.05	11
7	159.50	1	12.5	12.5	0.5	6.125	12.25	0.25	3.03125	12.125	0.1	1.205	12.05	12
8	172.00	1	13.5	13.5	0.5	6.625	13.25	0.25	3.28125	13.125	0.1	1.305	13.05	13
9	185.50	1	14.5	14.5	0.5	7.125	14.25	0.25	3.53125	14.125	0.1	1.405	14.05	14
10	200.00	1	15.5	15.5	0.5	7.625	15.25	0.25	3.78125	15.125	0.1	1.505	15.05	15
11	215.50	1	16.5	16.5	0.5	8.125	16.25	0.25	4.03125	16.125	0.1	1.605	16.05	16
12	232.00	1	17.5	17.5	0.5	8.625	17.25	0.25	4.28125	17.125	0.1	1.705	17.05	17
13	249.50	1	18.5	18.5	0.5	9.125	18.25	0.25	4.53125	18.125	0.1	1.805	18.05	18
14	268.00	1	19.5	19.5	0.5	9.625	19.25	0.25	4.78125	19.125	0.1	1.905	19.05	19
15	287.50	1	20.5	20.5	0.5	10.125	20.25	0.25	5.03125	20.125	0.1	2.005	20.05	20
16	308.00	1	21.5	21.5	0.5	10.625	21.25	0.25	5.28125	21.125	0.1	2.105	21.05	21
17	329.50	1	22.5	22.5	0.5	11.125	22.25	0.25	5.53125	22.125	0.1	2.205	22.05	22
18	352.00	1	23.5	23.5	0.5	11.625	23.25	0.25	5.78125	23.125	0.1	2.305	23.05	23
19	375.50	1	24.5	24.5	0.5	12.125	24.25	0.25	6.03125	24.125	0.1	2.405	24.05	24
20	400.00	1	-400	-400	0.5	12.625	25.25	0.25	6.28125	25.125	0.1	2.505	25.05	25

Table A-1c: Cubic Cost Function

output	cost	Δq	ΔC	ΔC/Δq	Δq	ΔC	ΔC/Δq	Δq	ΔC	ΔC/Δq	Δq	ΔC	ΔC/Δq	dC/dq
0	$100	1	9.05	9.05	0.5	$4.76	9.5125	0.25	$2.44	9.75313	0.1	$0.99	$9.90	$10.00
1	$109	1	7.35	7.35	0.5	$3.87	7.7375	0.25	$1.99	7.94063	0.1	$0.81	8.0655	$8.15
2	$116	1	5.95	5.95	0.5	$3.13	6.2625	0.25	$1.61	6.42812	0.1	$0.65	6.5305	$6.60
3	$122	1	4.85	4.85	0.5	$2.54	5.0875	0.25	$1.30	5.21563	0.1	$0.53	5.2955	$5.35
4	$127	1	4.05	4.05	0.5	$2.11	4.2125	0.25	$1.08	4.30312	0.1	$0.44	4.3605	$4.40
5	$131	1	3.55	3.55	0.5	$1.82	3.6375	0.25	$0.92	3.69062	0.1	$0.37	3.7255	$3.75
6	$135	1	3.35	3.35	0.5	$1.68	3.3625	0.25	$0.84	3.37812	0.1	$0.34	3.3905	$3.40
7	$138	1	3.45	3.45	0.5	$1.69	3.3875	0.25	$0.84	3.36563	0.1	$0.34	3.3555	$3.35
8	$142	1	3.85	3.85	0.5	$1.86	3.7125	0.25	$0.91	3.65313	0.1	$0.36	3.6205	$3.60
9	$145	1	4.55	4.55	0.5	$2.17	4.3375	0.25	$1.06	4.24063	0.1	$0.42	4.1855	$4.15
10	$150	1	5.55	5.55	0.5	$2.63	5.2625	0.25	$1.28	5.12812	0.1	$0.51	5.0505	$5.00
11	$156	1	6.85	6.85	0.5	$3.24	6.4875	0.25	$1.58	6.31562	0.1	$0.62	6.2155	$6.15
12	$162	1	8.45	8.45	0.5	$4.01	8.0125	0.25	$1.95	7.80313	0.1	$0.77	7.6805	$7.60
13	$171	1	10.35	10.35	0.5	$4.92	9.8375	0.25	$2.40	9.59062	0.1	$0.94	9.4455	$9.35
14	$181	1	12.55	12.55	0.5	$5.98	11.963	0.25	$2.92	11.6781	0.1	$1.15	11.511	$11.40
15	$194	1	15.05	15.05	0.5	$7.19	14.388	0.25	$3.52	14.0656	0.1	$1.39	13.876	$13.75
16	$209	1	17.85	17.85	0.5	$8.56	17.113	0.25	$4.19	16.7531	0.1	$1.65	16.541	$16.40
17	$227	1	20.95	20.95	0.5	$10.07	20.138	0.25	$4.94	19.7406	0.1	$1.95	19.506	$19.35
18	$248	1	24.35	24.35	0.5	$11.73	23.463	0.25	$5.76	23.0281	0.1	$2.28	22.771	$22.60
19	$272	1	28.05	28.05	0.5	$13.54	27.088	0.25	$6.65	26.6156	0.1	$2.63	26.336	$26.15
20	$300	1	-300	-300	0.5	$15.51	31.013	0.25	$7.63	30.5031	0.1	$3.02	30.201	$30.00

Basic Rules of Differentiation

Since the purpose of this appendix is limited to giving the uninitiated student a feel for calculus, we will not go to the trouble of rigorously proving all of the rules. I will develop the first few mathematically so that you will not be overly shocked if you do decide to enroll in a calculus course. The first, and probably most useful rule for economics is the **polynomial rule**. The basic idea of a derivative is the function which is derived from another function, whose domain is the values of the independent variables and whose range is the **slope** of the parent function at that value for the independent variable. We compute the derivative as shown in Tables A-1a, b, and c: we divide the change in the dependent variable by the change in the independent variable, as we allow the change in the independent variable to get progressively smaller (approach zero). A simple polynomial function follows the general form

where *c* is some constant, *x* is the value of the independent variable, and *n* is an integer.

If we let Δx be the change in *x*, the change in *y* is $c(x + \Delta x)^n - cx^n$. When we expand the first term, we get the following:

$$c(x + \Delta x)^n = cx^n + cnx^{n-1}(\Delta x) + \cdots + cnx(\Delta x)^{n-1} + c(\Delta x)^n.$$

When we subtract cx^n from this expression, the first term disappears; all the remaining terms contain the variable Δx. As long as $\Delta x \neq 0$, we can divide through and eliminate Δx from the numerator. Then we imagine that Δx goes all the way to zero, and we conclude:

=

$=cnx^{n-1}$.

Specific applications for this simple polynomial rule are relevant to economics. First, suppose that the function is a **constant function**. For instance, fixed cost is invariant with respect to output:

$$FC = c = cq^0$$

Applying the rule leads to the common-sense result, . As output changes, fixed cost does not change. We have already noted that a linear function has a constant slope. If output cost function has the form, $C = bq$, application of the rule implies

Finally, if a constant function has the form $C = cq^n$, we say that the cost function is a simple polynomial of order n; the resulting marginal cost function will have order n-1: .

If $n < 0$, x^n is the reciprocal of x^{-n}; if $n = -2$, . For instance, consider the firm's average fixed cost function, *AFC*:

.

If n is a fraction, the rule still holds. Let . Even if n is an irrational (or transcendental) number, the rule holds: If $y = x^{\pi}$; , where π is the ratio of the circumference of a circle to its diameter.

One of the driving forces in science, and therefore in mathematics and economics, is to *fill in the gaps*. Reversing the process of *differential* calculus is *integral* calculus – literally, integrating from the derivative to its parent function. For instance, if the derivative of a function is 2x, the original function has the form $y = c + x^2$ (since the derivative of a quadratic function is a linear function, and the derivative of a constant function is zero). So what function has as its derivative x^{-1} = ? Mathematicians, by means too complex to be explored here, discovered that the natural logarithm of a number had as its derivative the reciprocal of that number. If $x = e^y$,[2] then $y = \log_e y = \ln x$, and

We can expand our simple polynomial function to a typical polynomial function of the form

,

by using the **addition rule**: When a function can be expressed as the sum of simpler functions. A typical polynomial function is the sum of a series of terms, starting with the constant a, followed by a linear term, then a quadratic term, and so forth to any integer value n. The rank of the polynomial is determined by the term with the largest exponent whose constant term is not 0. According to the addition rule, the derivative of a sum of functions is the sum of the derivatives of those functions:

If $f(x) = g(x) + h(x)$, .

Some important applications of the addition rule apply to profit, which is the difference between the revenue and the cost that results from producing a particular rate of output.

[2] The natural number e is the base of the natural logarithms and is an irrational number whose approximate value is $e \approx 2.718281$. This number is derived by taking the limit of as n approaches infinity. As a variable grows over time, e^t predicts the change in the variable after t periods, when that growth is continuously compounded. The **exponential function** has applications in economic growth theories.

— — — .

The change in profit is marginal revenue minus marginal cost.

In microeconomics we study many relationships that are **recursive** relationships between several functions. For instance, a firm hires labor to produce output, which it then sells to generate revenue. The change in revenue due to hiring an additional worker (the worker's marginal revenue product) is the change in revenue due to the change in output, times the change in output due to the change in labor. The **chain rule** states this rule formally as:

Let $R = R(q)$ and $q = f(L)$, then — — —

Specifically, suppose the revenue function for a price-taker is $R = 10q - q^2$; and $q = 25L - L^2$. There are two ways to compute the marginal revenue product of labor. One way is to substitute the production function for q and then to take the derivative: $R = 10(25L - L^2) = 250L - 10L^2 \rightarrow$ — . The other approach is to apply the chain rule: —=— — . When we have two options, economics tells us to pick the best one.

Suppose that a function is generated as the product of two other functions. For instance, a monopolist's revenue is the product of price (which itself varies with output) and output itself. How does revenue change as output increases? What is the derivative of a function that is the product of two other functions? We might be tempted to conclude that the derivative would be the product of the two derivatives; but this would be wrong. An illustration will show why.

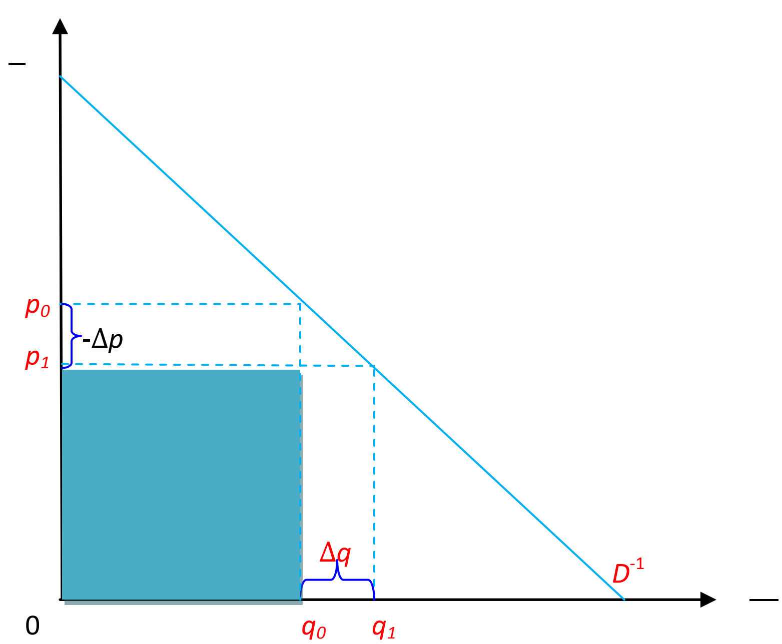


Figure A-2: Price Change and Marginal Revenue

When the firm sells q_0 units at price p_0 for revenue of $R_0 = p_0q_0$. When output increases to q_1 the price falls to p_1 and the new revenue is p_1q_1. Did the output increase and resulting price decrease cause revenue to increase, decrease, or remain the same? The answer is *yes*, which is not particularly useful. Technically, we want to know how rectangle p_1q_1 compares with rectangle p_0q_0. In Figure A-2 the two rectangles have the blue area, p_1q_0 in common; so that the issue translates into a comparison between $p_1(\Delta q)$ – the revenue gain from the extra units sold – and $q_0(-\Delta p)$, the revenue loss due to the price cut. By definition:

The derivative of the product of two functions equals the sum of each function times the derivative of the other function. If $y = f(x) \times g(x)$:

Each function behaves as a *scale* (i.e., a multiplicative constant) when the effect of the change in the other function is being determined. We first find the effect of the change in $g(x)$, given the value of $f(x)$, and then add the effect of the change of $f(x)$, given the value of $g(x)$.

We have reached the limit of intuition and must instead rely on logic when we confront the **quotient rule**. If we are given a function of the form take the bottom times the derivative of the top minus the top times the derivative of the bottom, over the bottom squared:

If there weren't practical reasons for computing the derivative of the ratio of two functions, we might be tempted to leave such counterintuitive calculus rules to the mathematicians. But consider average cost: it is equal to total cost, which is a function of output, to output (which is a function of itself). Understanding how average cost changes as output changes is key to understanding long-run competitive equilibrium – a central concept in microeconomic theory.

Using concepts we have already developed – the chain rule and the product rule – we can derive the quotient rule. Once we are able to derive a rule, we need not worry about memorizing it. The function is equivalent to $y = f(x)\text{x}[g(x)]^{-1}$, where $[g(x)]^{-1}$ is clearly a function of a function. Applying the product rule:

(by chain rule)

(factoring out)

One final concept will establish the rules that will guide you through the text. As long as a derivative is a smooth (continuous) function is itself a continuous function, it also has a derivative. We define the **second derivative** of a function as the result of taking the derivative of a derivative – that is, the effect of taking the derivative of a function two times. Suppose that the short-run production function is given by $q = 25L - L^2$; the domain of this function is $0 \leq L \leq 25$, since if $L > 25$, the predicted rate of output is negative, which is clearly impossible. The first derivative is the marginal product of labor:

=

The first derivative is positive for the first 12.5 labor hours, and then is negative between 12.5 and 25 labor hours; output is maximized when L = 12.5, if the firm continued to hire labor, the last 12.5 labor hours would destroy the output of the first 12.5 labor hours. An important concept in production theory is the law of diminishing marginal productivity: as more workers are hired, the additions to output (labor's marginal product) eventually gets smaller. This rule implies that the *derivative of marginal product with respect to labor, which is the second derivative of output with respect to labor, is negative.* For reasons lost in mathematical antiquity,[3] we express the second derivative as:

.

A convention that we will also use is use a single apostrophe for the first derivative and a double apostrophe or quotation mark to express a second derivative:

.

Summary of Differentiation Rules

1. If $y = cx^n$,
2. If $y = ln(x) = log_e x$; =
3. If $y = f(x)$ and $x = g(w)$;
4. If $y = f(x) \pm g(x)$, .
5. If $y = f(x)g(x)$, $= f(x)$
6. If .
7. $f''(x) =$

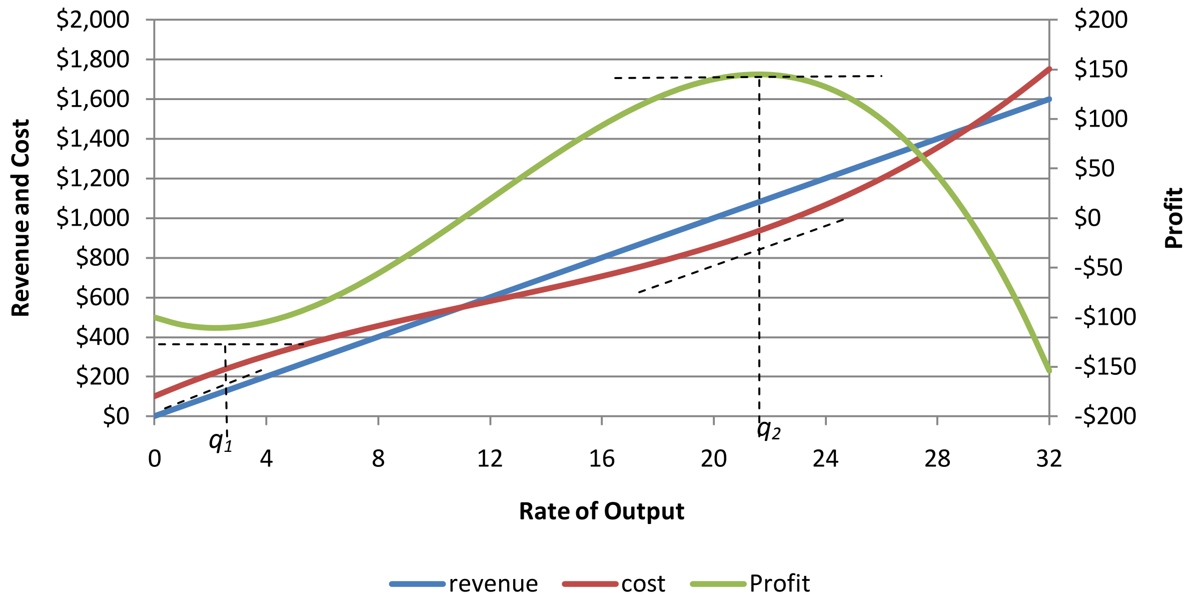
Calculus and Optimization

As you are aware from your principles of microeconomics course, and from our earlier discussion, the principle use of calculus in microeconomic theory is finding the maximum or minimum of a relevant dependent variable. The most obvious example is the firm's attempt to maximize profit. In Figure A-3 I plot the firm's rate of output on the horizontal axis, and its revenue (a linear function of output), its costs (a cubic function of output), and its profit (revenue minus cost). To make it easier to discern the relation between profit and output, the profit function is plotted on the right vertical axis. Where the profit function reaches its highest point, at q_2, and its lowest point, q_1, the slope of the profit function is zero. What distinguishes the maximum from the minimum is determined by the behavior of the function in the neighborhood of the "flat place." At q_1, the slope of the profit function changes from negative (below q_1) to zero (at q_1), to positive (above q_1). When a function reaches a *minimum* point, the first derivative is zero (the function is flat) and the second derivative is positive (the slope is increasing).[4]

[3] Actually, I don't know and don't care to do the research!

[4] A particularly telling application of this concept involves the aborted attempt to introduce microeconomic efficiency into central planning in the Soviet Union. As the story goes, Soviet economists noted that Karl Marx actually had little to say about how a post-capitalist economy should be organized. One economist named Lieberman sug-

Figure A-3:
Firm's Revenue, Cost and Profit



At a maximum point, like q_2, the first derivative is zero (the curve is flat), and in the neighborhood of q_2 the slope goes from positive (below q_2), to zero (at q_2), to negative (above q_2). At a local maximum the first derivative is zero and the second derivative is negative (the slope changes from positive to zero to negative as it "tops out.")

Another feature of Figure A-3 worth noticing is that at both q_1 and q_2, the revenue function has the same slope as the cost function. This reflects an important microeconomic theory: the firm maximizes profit by producing that rate of output where marginal cost equals marginal revenue. What Figure A-3 reminds us is that if the firm merely sets marginal cost equal to marginal revenue, it runs the risk of producing the *profit minimum* instead of the profit maximum. The **second order condition** requires that — ; this requirement is satisfied when marginal revenue is decreasing faster than marginal cost at the critical rate of output. A sufficient condition is that marginal cost curve is positively sloped, although we will see that it is possible for a natural monopoly to maximize profit when both marginal revenue and marginal cost are negatively sloped, as long as marginal revenue is decreasing faster than marginal cost is.

Partial Derivatives: Never Having to Say *Ceteris Paribus*

Microeconomic theory took a dramatic turnin the 1870's when William Stanley Jevons in Great Britain, Carl Menger in Germany, and Leon Walras in Austria apparently resolved the *diamond-water*

gested that Soviet plant managers follow the rule "set output where marginal cost equals price. Being unwilling to question the wisdom of orders from higher up, plant managers dutifully expanded output as long as marginal cost exceeded (the State determined) price, and ceased expanding output when marginal cost equaled price. By ignoring the "second order condition" Soviet plant managers minimized economic efficiency.

paradox posed by Adam Smith, by distinguishing between the first and second derivatives of a utility function with respect to the consumption of a particular good. Nearly a century earlier Adam Smith noted that few commodities were more useful than water, while most people went through their entire lives without ever seeing a diamond, let alone possessing one. It would seem a paradox, then, that diamonds are so much more valuable than water is, at least under normal circumstances. Smith understood that the answer to the paradox was scarcity; how to measure scarcity independently of value led him to his labor theory of value, an idea that foundered on the non sequiturs of Marxism.

The paradox is resolved by noting that while usefulness is a reflection of total utility, price, or value in exchange, (what) is a function of *marginal* utility. Marginal utility is the change in satisfaction due to a one unit increase in the consumption of one commodity, the consumption of all other commodities held constant. In order to measure the effect of one of many independent variables on satisfaction, economists use the concept of *ceteris paribus* – other factors remaining constant. Mathematics has an even more useful concept – the partial derivative.

Suppose that a household's satisfaction results from the consumption of many different commodities, measured as q_1 through q_n. The marginal utility of good 1 is the change in utility due to a one unit increase in the consumption of good 1, all other goods remaining constant. We can obtain this definition by modifying our definition of the derivative to treat other independent variables as constants:

$$MU_i =$$

A simple illustration may help. Suppose that $y = xz$, where x and z are independent variables, meaning that one variable can change without *causing* the other to change; if both variables should happen to change at the same time, those changes would be *coincidental*. Analytically, we could investigate the effect of a change of x on y by simply treating variable z as a constant. It follows that:

$$= \quad .$$

In this function, z acts as a scalar for x and x acts as a scalar for z. To get the value of y, we multiply the value of x by the value of z. If we want to determine the effect of a change of x, we treat z as a constant. The greater the value of z, the stronger the effect of a change in x will be on y. By contrast, suppose that $y = x + z$. In this case, each unit change in x causes a one unit change in y; the value of z helps determine the value of y, but it has no impact on the value of , which we can prove by the addition rule:

$$.$$

Partial derivatives are themselves functions with multiple independent variables, so that second and even higher order partial derivatives can be defined. The law of diminishing marginal utility is one such notion: Given $U = U(q_1,q_2,\ldots,q_n)$, $MU_i =$; diminishing marginal utility implies . This is mathematical expression for the statement "as the consumption of good *i* increases, its marginal utility decreases, the consumption of other commodities remaining constant."

With partial derivatives we can also express how changes in one variable influence the partial derivative of some other variable. For instance, if an increase in the consumption of one commodity increases the marginal utility of some other commodity, those two commodities are considered **complements**. For instance, let good 1 be pretzels and good 2 be mustard; having more mustard increases the satisfaction I get from pretzels:

$MU_{12} =$ (goods 1 and 2 are complements)

On the other hand, if an increase in the consumption of good #2 decreases the satisfaction obtained from an extra unit of good #1, those two goods are substitutes.

$MU_{12} =$ (goods 1 and 2 are substitutes).

Economists use *ceteris paribus* analysis to make complex analyses simpler. It is comforting to learn one additional lesson of partial derivatives: when we keep our changes in independent variables small, we can determine the sum of the effects of several variables changing at once through the **total differential,** which measures that total effect as sum the partial derivative of each independent variable times the change in that variable:

.

Suppose that an economist estimates a demand function of the form $Q_d = D(p_1,p_2,\ldots,p_n,y,t)$, which contains the prices of *n* goods, income (*y*), and some proxy for tastes, *t*. With sufficient data and foresight, that economist could predict the change in quantity demanded as:

$dQ_d =$.

Afterward/Forward

The purpose of this appendix has not been to intimidate but to enlighten. I want to emphasize that this is an economics course and this text is an economics book. This is not a mathematics course and this is not a mathematics book. Calculus and microeconomic theory are not substitutes, but complements. Professional economists find that mathematics helps them apply and verify economic theory. You could ignore the math and accept the economics on faith; you could, but it would be wrong.